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Convective amplification of real simple sources

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This paper examines the problem of a pulsating compact body moving with constant velocity. This problem is usually incorrectly treated as a convected monopole. The analysis here shows that the motion of such a 'real' source introduces additional coupled multipoles, whose combined effects generate previously unexpected convective features. The amplification obtained is not the monopole convective amplification $(1-M_r)^{-2}$. It is found to depend on the virtual mass tensor of the body, the minimum effect being $(1-M_r)^{-3}$. There is also amplification in the direction perpendicular to the flight path (unless the motion is parallel to a principal axis of the virtual mass tensor).

The field produced by oscillation of a convected compact body of constant geometry is also investigated. Again, this problem is often misrepresented as a moving dipole. Here it is shown that the effect of convection on such a real source is surprising and complicated. It cannot be described completely by Doppler factors, and there is amplification in the direction perpendicular to the source motion.

These two model problems serve as a warning that the effect of flight on real sources cannot be anticipated until such real sources are correctly modelled, and that also the influence of source motion is likely to be much greater than has so far been anticipated.

1. Introduction

The field produced by a moving monopole source is well known (Lienard & Wiechert 1900). Morse & Ingard (1968) assume that such an acoustic field may be produced by a pulsating compact body. That is incorrect.

Convective effects are extremely important in the aeronautical noise problem, where they are known (Hoch & Hawkins 1973) to be both significant and perplexing. Of course, there the precise nature of the sources is usually not known while prediction of flight changes in the acoustic field is invariably based on known results for mathematically convenient source models. That is how the apparently simple problem of a convected pulsating spherical source is so often treated incorrectly.

This paper examines two of the simplest problems in greater depth and shows that the consequences of *real* source motion are rather unexpected. Motion introduces additional coupled multipoles whose combined effects generate previously unexpected convective features. We examine a small deformable body whose centroid is in uniform motion and show that that motion amplifies the linear sound field by a factor $(1+\alpha_{ij}M_j\hat{x}_i)(1-M\cos\theta)^{-3}$, α_{ij} being the virtual mass tensor, $\mathbf{M}$ the convection velocity normalized to the speed of sound and $\hat{x}_i$ the direction cosine of the path taken by the sound travelling towards the distant observation point at an angle θ to $\mathbf{M}$, the direction of flight. The simplest source, that generated by a volume pulsation, does not receive the monopole convective amplification factor $(1-M\cos\theta)^{-2}$. The product $M_i\hat{x}_i$ or $M\cos\theta$ appears frequently in what follows and we shall write M_r in its place, the suffix implying the direction in which the particular sound ray is travelling to the distant field. The minimum effect $(1-M\cos\theta)^{-3}$ pertains when the virtual mass is zero, in accordance with the result found by Ffowcs Williams & Lovely (1975). Finite virtual mass both increases the magnitude of convective amplification and radically changes it in that there is also an amplification in the direction perpendicular to the flight path (unless the motion is parallel to a principal axis of the virtual mass tensor).

We also study the far sound field produced by an undeformable body moving with a small vibration superimposed on otherwise steady low Mach number convection. This sound field has previously been assumed (incorrectly) to be modelled by a moving dipole, see for example Ffowcs Williams & Hawkings (1969, equation 7.5). The effect of convection is then completely described by the multiplying factor $(1-M\cos\theta)^{-2}$. However, closer investigation shows that the effect of motion is far more complicated. It does not just involve Doppler factors, and there is amplification in a direction perpendicular to the source motion even for an isotropic body. For motion parallel to a principal axis of the inertial mass tensor, the non-convected field of a vibrating body is amplified in motion by the fourth power of the inverse Doppler factor, and in addition there is an omnidirectional term in the sound field!

These are significant differences not easily predictable from the substantial literature on the subject and give cause to expect that when the real complex sources of practical importance are effectively modelled some of the difficulties of accounting for flight effects on aircraft noise levels will be overcome.

2. The sound field produced by a weakly pulsating body in uniform motion

We consider the problem of a pulsating compact body moving with a constant velocity $\mathbf{U}$ through an inviscid fluid at rest. The Mach number $\mathbf{M}\,(=\mathbf{U}/c_0)$ is assumed small enough that M^2 is everywhere negligible in comparison with unity. The influence of motion is rather unexpected, so that we feel it prudent to demonstrate the result by three independent methods. In the first we select $P=(\rho/\rho_0-1)\,c_0^2+\frac{1}{2}u^2$ as the perturbed quantity, and derive the governing equation from a consideration of mass and momentum conservation as follows:

$$\frac{1}{\rho_0}\frac{\partial\rho}{\partial t}+\nabla.\mathbf{u}+\frac{\partial}{\partial x_i}\left(\frac{\rho-\rho_0}{\rho_0}u_i\right)=0,$$

$$\frac{\partial\mathbf{u}}{\partial t}+\nabla\left\{\tfrac{1}{2}u^2+\int\frac{dp}{\rho}\right\}-\mathbf{u}\wedge\boldsymbol{\omega}=0.$$

p, ρ, $\mathbf{u}$ and $\boldsymbol{\omega}$ are respectively the instantaneous values of pressure, density, velocity and vorticity, and the suffix zero implies the mean value in the distant field.

These two equations are combined to give an inhomogeneous acoustic wave equation valid throughout the fluid:

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\left(\tfrac{1}{2}u^2+\frac{\rho-\rho_0}{\rho_0}c_0^2\right) = -\nabla.(\mathbf{u}\wedge\boldsymbol{\omega})+\nabla^2\left\{\int\frac{dp}{\rho}-c_0^2\frac{\rho-\rho_0}{\rho_0}\right\} + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\tfrac{1}{2}u^2-\frac{\partial^2}{\partial x_i\,\partial t}\left(\frac{\rho-\rho_0}{\rho_0}u_i\right). \quad (1)$$

c_0 is a constant, which we choose to equal the acoustic wave speed at infinity.

In the appendix it is shown that

$$4\pi H(\mathbf{x},t)\,P(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\int_V\left[\frac{(\mathbf{u}\wedge\boldsymbol{\omega})_i}{r(1-M_r)}\right]d^3\boldsymbol{\eta}-\frac{\partial^2}{\partial t\,\partial x_i}\int_V\left[\frac{(\rho/\rho_0-1)\,u_i}{r(1-M_r)}\right]d^3\boldsymbol{\eta}$$
$$+\int_V\left[\frac{\nabla^2\{\int\rho^{-1}dp-(\rho/\rho_0-1)\,c_0^2\}}{r(1-M_r)}\right]d^3\boldsymbol{\eta}+\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_V\left[\frac{\tfrac{1}{2}u^2}{r(1-M_r)}\right]d^3\boldsymbol{\eta}$$
$$-\frac{\partial}{\partial x_i}\int_S\left[\frac{Pn_i}{r(1-M_r)}\right]dS(\boldsymbol{\eta})+\frac{\partial}{\partial t}\int_S\left[\frac{(u_i-U_i)\,n_i(\rho/\rho_0-1)}{r(1-M_r)}\right]dS(\boldsymbol{\eta})$$
$$+\int_S\left[\frac{-\partial P/\partial y_i+(\mathbf{u}\wedge\boldsymbol{\omega})_i-U_i\nabla.\mathbf{u}}{r(1-M_r)}\right]dS(\boldsymbol{\eta}), \quad (2)$$

where n_i is the direction cosine of the normal pointing into the fluid at S, a surface of fixed shape which completely encloses the body at all times (and possibly some fluid too), and moves with the mean velocity of the body. V is the region outside S. H is the Heaviside function equal to unity in V and zero otherwise. The equation is exact and all the above flow quantities vary with position and time.

The origin of the co-ordinate $\boldsymbol{\eta} = \mathbf{y}-\mathbf{U}t$ moves uniformly with the centroid of the pulsating body, and the square brackets mean that the function they enclose is to be evaluated at retarded time $t-r/c$, r being the distance travelled by the wave from its source at $\mathbf{y}$ to the field point $\mathbf{x}$, i.e. $r = |\mathbf{x}-\mathbf{y}|$.

For low Mach number flow about a compact body, we can neglect terms quadratic in the density perturbation, so that

$$p-p_0 = c_0^2(\rho-\rho_0), \quad \int\rho^{-1}dp = c_0^2(\rho/\rho_0-1).$$

The analysis can in fact be continued without neglecting the quadratic density fluctuation terms if one assumes a definite functional relation between the pressure and density fields, p/ρ^γ = constant, for example. Then the density can be determined once the pressure has been calculated from an incompressible modelling of the near-field flow. We have done that calculation to ensure that the error incurred in the neglect of the quadratic density fluctuation terms is smaller by at least the compactness ratio or the square of the flow Mach number than the terms we retain. That error is negligible in the compact body, low Mach number limit, so that (2) may be written as

$$
\begin{aligned}
4\pi HP(\mathbf{x},t) = & -\frac{\partial}{\partial x_i}\int_V \left[\frac{(\mathbf{u}\wedge\boldsymbol{\omega})_i}{r(1-M_r)}\right] d^3\boldsymbol{\eta} - \frac{\partial^2}{\partial t\,\partial x_i}\int_V \left[\frac{u_i(p-p_0)/(\rho_0 c_0^2)}{r(1-M_r)}\right] d^3\boldsymbol{\eta} \\
& + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_V \left[\frac{\frac{1}{2}u^2}{r(1-M_r)}\right] d^3\boldsymbol{\eta} - \frac{\partial}{\partial x_i}\int_S \left[\frac{Pn_i}{r(1-M_r)}\right] dS(\boldsymbol{\eta}) \\
& + \frac{\partial}{\partial t}\int_S \left[\frac{(u_i-U_i)\,n_i(p-p_0)/(\rho_0 c_0^2)}{r(1-M_r)}\right] dS(\boldsymbol{\eta}) \\
& + \int_S \left[\frac{-\partial P/\partial y_i + (\mathbf{u}\wedge\boldsymbol{\omega})_i - U_i\nabla\cdot\mathbf{u}}{r(1-M_r)}\right] n_i\, dS(\boldsymbol{\eta}),
\end{aligned}
\tag{3}
$$

with $P = (p-p_0)/\rho_0 + \frac{1}{2}u^2$. This equation is in a suitable form to apply the hypothesis underlying the Lighthill theory of aerodynamic sound; the sound field itself is unimportant as a wave generator, and the various integrands on the right-hand side of (3) may be defined as sound sources.

Howe (1975) has shown that, when the flow variables in the source terms of (3) are replaced by their values in incompressible flow satisfying the same boundary conditions, an error of order M^2 in comparison with unity is made. This we neglect. Thus, in order to determine these source terms, we consider here the problem of a pulsating body moving through an incompressible inviscid fluid at rest. From Kelvin's circulation theorem we see that the flow is at all times irrotational and a potential ϕ exists such that $\mathbf{u} = \nabla\phi$, with $\nabla^2\phi = 0$. Then Bernoulli's equation gives

$$
-\partial\phi/\partial t = \tfrac{1}{2}u^2 + (p-p_0)/\rho_0,
$$

and (3) becomes

$$
\begin{aligned}
4\pi H(\mathbf{x})\,P(\mathbf{x},t) = & \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_V \left[\frac{\frac{1}{2}u^2}{r(1-M_r)}\right] d^3\boldsymbol{\eta} + \frac{\partial}{\partial x_i}\int_S \left[\frac{\partial\phi/\partial t}{r(1-M_r)}\right] n_i\, dS(\boldsymbol{\eta}) \\
& + \int_S \left[\frac{\partial^2\phi/\partial y_i\,\partial t}{r(1-M_r)}\right] n_i\, dS(\boldsymbol{\eta}) + Q(\mathbf{x},t),
\end{aligned}
\tag{4}
$$

where

$$
Q(\mathbf{x},t) = -\frac{\partial^2}{\partial t\,\partial x_i}\int_V \left[\frac{u_i(p-p_0)/(\rho_0 c_0^2)}{r(1-M_r)}\right] d^3\boldsymbol{\eta} + \frac{\partial}{\partial t}\int_S \left[\frac{(u_i-U_i)\,n_i(p-p_0)/(\rho_0 c_0^2)}{r(1-M_r)}\right] dS(\boldsymbol{\eta}).
$$

We shall see later that the terms in Q do not give rise to a significant sound field.

We shall now proceed to evaluate these source terms, to determine the sound radiated by the 'weak pulsation of a compact body', and the effect of motion on that sound.

We consider a situation where the body has a fixed shape but variable scale. We denote the length of a reference line by $A(t)$ and define

$$
v(t) = dA/dt, \quad a = A(O).
$$

We restrict the analysis to a small slow pulsation, by which we mean a pulsation described by $v = \epsilon^2 c_0 V$, where $V = O(1)$ and ϵ is a small parameter. This ensures that the surface normal velocity is always entirely subsonic. Also we require that

$$
\int^T V(T')\,dT', \frac{dV}{dT}, \frac{d^2V}{dT^2}, \dots = O(1),
$$

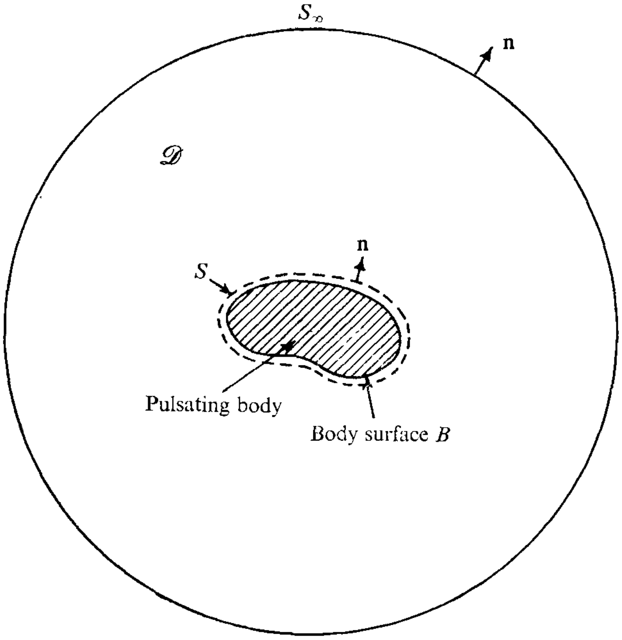

FIGURE 1. The surfaces over which integrals are evaluated in the pulsating body case.

T being a non-dimensional slow time defined by $T = \epsilon c_0 a^{-1} t$. This is a compactness condition. Then $v = \epsilon^2 c_0 V$ gives

$$dv/dt = \epsilon^3 c_0^2 a^{-1} dV/dT, \quad d^2v/dt^2 = O(\epsilon^4),$$

and also

$$A(t) = a + \int_0^t v(t')\,dt' = a + \epsilon a \int_0^T V(T')\,dT'$$
$$= a + O(\epsilon a) < a + \epsilon a k, \quad \text{say}.$$

S may be taken to be a surface with the same shape as the body, with the length of its reference line being $a + \epsilon k a$; see figure 1.

From this it is evident that we imply by the term 'weak pulsation of a compact body' that the surface vibrational velocity is always much less than c_0 and that waves travel a distance much greater than a in one unit of time that characterizes the surface motion. Furthermore, we require that the motion is even more subsonic than the body is compact!

The velocity potential ϕ satisfies $\nabla^2\phi = 0$ in the fluid, with the boundary conditions $\mathbf{n}.\nabla\phi = vg(\boldsymbol{\eta}) + \mathbf{U}.\mathbf{n}$ on the surface of the body and

$$\nabla\phi \to 0 \quad \text{as} \quad \boldsymbol{\eta} \to \infty;$$

$g(\boldsymbol{\eta})$ is a function of geometry, whose precise form is actually unimportant in the analysis that follows.

Because of the linearity of the equation and the boundary conditions, we can consider this as a superposition of two problems and write

$$\phi = v\Psi(\boldsymbol{\eta}, A) + \mathbf{U}.\boldsymbol{\Phi}(\boldsymbol{\eta}, A),$$

where Ψ and $\mathbf{\Phi}$ are functions of position and time that depend on the shape and the instantaneous size of the body. This is a generalization of the procedure described by Batchelor (1970, p. 402).

It is convenient to write the dependence of $\mathbf{\Phi}$ on the size of the body explicitly. In order to do this we non-dimensionalize lengths with respect to A, and, writing $\boldsymbol{\xi} = A^{-1}\boldsymbol{\eta}$, we introduce a non-dimensional function $\tilde{\mathbf{\Phi}}(\boldsymbol{\xi}, A)$ defined by

$$A\tilde{\mathbf{\Phi}}(\boldsymbol{\xi}, A) = \mathbf{\Phi}(\boldsymbol{\eta}, A);$$

then $\tilde{\mathbf{\Phi}}$ satisfies $\nabla^2_{\xi}(\mathbf{U}.\tilde{\mathbf{\Phi}}) = 0$. The boundary condition on the body gives

$$n_j\,\partial(\mathbf{U}.\tilde{\mathbf{\Phi}})/\partial\xi_j = \mathbf{U}.\mathbf{n}.$$

Thus $A(t)$ has been eliminated from the problem for $\tilde{\mathbf{\Phi}}$, so $\tilde{\mathbf{\Phi}}$ is necessarily a function only of the body shape. Therefore

$$\phi = v\Psi(\boldsymbol{\eta}, A) + A\mathbf{U}.\tilde{\mathbf{\Phi}}(A^{-1}\boldsymbol{\eta}),$$

so that
$$\partial\phi/\partial t|_{\mathbf{y}} = \dot{v}\Psi - v^2\,\partial\Psi/\partial A + v\Theta$$

and
$$\frac{\partial}{\partial y_i}\left(\frac{\partial\phi}{\partial t}\bigg|_{\mathbf{y}}\right) = \frac{\partial}{\partial\eta_i}\left(\frac{\partial\phi}{\partial t}\bigg|_{\mathbf{y}}\right) = \dot{v}\frac{\partial\Psi}{\partial\eta_i} - v^2\frac{\partial^2\Psi}{\partial\eta_i\,\partial A} + v\frac{\partial\Theta}{\partial\eta_i},$$

where
$$\Theta = -U_k\,\partial\Psi/\partial\eta_k + U_j\tilde{\Phi}_j + AU_j\,\partial\tilde{\Phi}_j/\partial A - U_jU_kA\,\partial\tilde{\Phi}_j/\partial\eta_k.$$

We now express v in terms of the non-dimensional form V in order to see the relative magnitude of these terms.

$$\frac{\partial\phi}{\partial t}\bigg|_{\mathbf{y}} = \epsilon^2c_0\,V\Theta + \epsilon^3c_0^2a^{-1}\frac{dV}{dT}\Psi + O(\epsilon^4)$$

and
$$\frac{\partial}{\partial y_i}\left(\frac{\partial\phi}{\partial t}\bigg|_{\mathbf{y}}\right) = \epsilon^2C_0\,V\frac{\partial\Theta}{\partial\eta_i} + \epsilon^3c_0^2a^{-1}\frac{dV}{dT}\frac{\partial\Psi}{\partial\eta_i} + O(\epsilon^4).$$

Since
$$\partial\Theta/\partial\tau|_{\eta} = \epsilon^2c_0\,V\,\partial\Theta/\partial A \tag{5a}$$

we have
$$\frac{\partial}{\partial\tau}\left(\frac{\partial\phi}{\partial t}\bigg|_{\mathbf{y}}\right)_{\eta} = \epsilon^3c_0^2a^{-1}\frac{dV}{dT}\Theta + O(\epsilon^4), \tag{5b}$$

$$\frac{\partial}{\partial\tau}\left(\frac{\partial^2\phi}{\partial y_i\,\partial t}\right)_{\eta} = \epsilon^3c_0^2a^{-1}\frac{dV}{dT}\frac{\partial\Theta}{\partial\eta_i} + O(\epsilon^4) \tag{5c}$$

and
$$\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial\phi}{\partial t}\right)_{\eta} = O(\epsilon^4), \quad \frac{\partial^2}{\partial\tau^2}\left(\frac{\partial^2\phi}{\partial y_i\,\partial t}\right) = O(\epsilon^4).$$

From (4),

$$4\pi HP(\mathbf{x}, t) = \int_S\left[\frac{-n_ix_i}{c_0r^2(1-M_r)^2}\frac{\partial}{\partial\tau}\left(\frac{\partial\phi}{\partial t}\right)_{\eta} + \frac{n_i}{r(1-M_r)}\frac{\partial}{\partial y_i}\left(\frac{\partial\phi}{\partial t}\right)\right]_{\tau^*}dS(\boldsymbol{\eta}) + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_V\left[\frac{\frac{1}{2}u^2}{r(1-M_r)}\right]d\boldsymbol{\eta} + Q(\mathbf{x}, t),$$

where the retarded time τ^* satisfies $c_0\tau^* = |\mathbf{x} - \mathbf{U}\tau^* - \boldsymbol{\eta}|$, which for $\mathbf{x}$ in the far field reduces to

$$\tau^* = \frac{1}{1-M_r}\left(t - \frac{|\mathbf{x}|}{c_0} + \frac{\mathbf{x}.\boldsymbol{\eta}}{|\mathbf{x}|\,c_0}\right).$$

We can expand the terms in the surface integral as a Taylor series in time, and from the order of magnitude of the time derivatives found in (5), only the first time derivatives are important. Hence

$$4\pi HP(\mathbf{x},t) = \int_S \left\{ \frac{n_i}{r(1-M_r)} \frac{\partial^2 \phi}{\partial y_i \partial t} + \frac{x_i}{r^2 c_0 (1-M_r)^2} \left(\eta_i n_j \frac{\partial}{\partial \tau} \left(\frac{\partial^2 \phi}{\partial y_i \partial t} \right)_{\eta} - n_i \frac{\partial}{\partial \tau} \left(\frac{\partial \phi}{\partial t} \right)_{\eta} \right) \right\}_{\tau_1} \times dS(\boldsymbol{\eta}) + O(\epsilon^4) + O(M^2 \epsilon^3),$$

where
$$\tau_1 = \frac{1}{1-M_r}\left(t - \frac{|\mathbf{x}|}{c_0}\right),$$
so that

$$4\pi HP(\mathbf{x},t) = \frac{\epsilon^2 c_0}{r(1-M_r)} \int_S \left\{ V \frac{\partial \Theta}{\partial \eta_i} + \frac{\epsilon c_0}{a} \frac{dV}{dT} \frac{\partial \Psi}{\partial \eta_i} \right\}_{\tau_1} n_i \, dS(\boldsymbol{\eta}) + \frac{x_i \epsilon^3 c_0}{r^2 (1-M_r)^2 a} \frac{dV}{dT} \int_S \left\{ \eta_i n_j \frac{\partial \Theta}{\partial \eta_j} - n_i \Theta \right\}_{\tau_1} dS(\boldsymbol{\eta}). \quad (6)$$

Now Ψ and $\tilde{\Phi}_j$ are well-defined harmonic functions which are determined by the boundary conditions on the body. In the region exterior to the body, Ψ and $\tilde{\Phi}_j$ may be expressed as a sum of harmonic functions:

$$\Psi = \frac{b}{\eta} + b_i \frac{\partial}{\partial \eta_i}\left(\frac{1}{\eta}\right) + b_{ij} \frac{\partial^2}{\partial \eta_i \partial \eta_j}\left(\frac{1}{\eta}\right) + \dots,$$
$$\tilde{\Phi}_j = \frac{Ac_j}{\eta} + A^2 c_{jk} \frac{\partial}{\partial \eta_k}\left(\frac{1}{\eta}\right) + \dots,$$

where the c's are independent of A, and $\eta = |\boldsymbol{\eta}|$. It is well known from potential theory (e.g. Batchelor 1970) that the velocity potential ϕ outside a body is such that $\phi = -m(t)/(4\pi\eta) + a_{ij}\eta_j/\eta^3 +$ higher harmonics, where $m(t)$ is the rate of change of volume of the body.

If the volume of a body with unit reference length is V_0 then a body with length scale A has volume $V_0 A^3$, and the rate of change of volume is $3V_0 A^2 v$. Hence, since this is independent of $\mathbf{U}$, c_j is zero and

$$\left.\begin{aligned} \Psi &= -3V_0 A^2/4\pi\eta + \text{higher harmonics}, \\ \tilde{\Phi}_j &= A^2 c_{jk}\, \partial\eta^{-1}/\partial\eta_k + \text{higher harmonics}. \end{aligned}\right\} \quad (7)$$

We see that Φ_j and Ψ are well-defined functions (for $\boldsymbol{\eta} \neq 0$). The integrals over the surface S in (6) may be evaluated by applying the divergence theorem to a region $\mathscr{D}$ between S and a large sphere S_∞.

Of course this step is purely formal and carries no implication that the harmonic functions represent the actual distant field. Then, with the normals to the surfaces in the directions sketched in figure 1,

$$\frac{\epsilon^2 c_0}{r(1-M_r)} \int_S \left\{ V \frac{\partial \Theta}{\partial \eta_i} + \frac{\epsilon c_0}{a} \frac{dV}{dT} \frac{\partial \Psi}{\partial \eta_i} \right\} n_i \, dS(\boldsymbol{\eta}) + \frac{x_i}{r^2 (1-M_r)^2} \frac{\epsilon^3 c_0}{a} \frac{dV}{dT} \int_S \left\{ \eta_i n_j \frac{\partial \Theta}{\partial \eta_j} - n_i \Theta \right\} dS(\boldsymbol{\eta})$$
$$= \frac{\epsilon^2 c_0}{r(1-M_r)} \int_{S_\infty} \left\{ V \frac{\partial \Theta}{\partial \eta_i} + \frac{\epsilon c_0}{a} \frac{dV}{dT} \frac{\partial \Psi}{\partial \eta_i} \right\} n_i \, dS(\boldsymbol{\eta}) + \frac{x_i}{r^2 (1-M_r)^2} \frac{\epsilon^3 c_0}{a} \frac{dV}{dT} \int_{S_\infty} \left\{ \eta_i n_j \frac{\partial \Theta}{\partial \eta_i} - n_i \Theta \right\} dS(\boldsymbol{\eta}) \quad (8)$$

since $$\nabla^2\Theta = \nabla^2\Psi' = 0, \quad \frac{\partial}{\partial \eta_j}\left(\eta_i \frac{\partial \Theta}{\partial \eta_j}\right) = \frac{\partial \Theta}{\partial \eta_i}.$$

Expression (8) shows that only the first terms in Ψ' and Φ_j contribute to the acoustic field. We can now simply evaluate these integrals using (7).

$$\frac{\epsilon^2 c_0}{r(1-M_r)} \int_{S\infty} \left\{ V \frac{\partial \Theta}{\partial \eta_i} + \frac{\epsilon c_0}{a} \frac{dV}{dT} \frac{\partial \Psi'}{\partial \eta_i} \right\} n_i \, dS(\boldsymbol{\eta}) = \frac{\epsilon^3 c_0^2}{a} \frac{dV}{dT} \frac{3V_0 A^2}{r(1-M_r)}$$
$$= \epsilon^3 c_0^2 a \frac{dV}{dT} \frac{3V_0}{r(1-M_r)} + O(\epsilon^4). \quad (9)$$

Now $A\,\partial\tilde{\boldsymbol{\Phi}}/\partial A = 2\tilde{\boldsymbol{\Phi}} + O(\eta^{-3})$ from (7) and so

$$\int_{S\infty} \left\{ \eta_i n_j \frac{\partial \Theta}{\partial \eta_j} - n_i \Theta \right\} dS = U_k \int_{S\infty} \left\{ n_i \frac{\partial \Psi'}{\partial \eta_k} - \eta_i n_j \frac{\partial^2 \Psi'}{\partial \eta_j \partial \eta_k} + 3\left(\eta_i n_j \frac{\partial \tilde{\Phi}_k}{\partial \eta_j} - n_i \tilde{\Phi}_k \right) \right\} dS(\boldsymbol{\eta}).$$

Now

$$\int_{S\infty} \left(n_i \frac{\partial \Psi'}{\partial \eta_k} - \eta_i n_j \frac{\partial^2 \Psi'}{\partial \eta_j \partial \eta_k} \right) dS$$
$$= \frac{3V_0 A^2}{4\pi} \int_{S\infty} \left(\frac{n_i \eta_k}{\eta^3} - \frac{\eta_i n_k}{\eta^3} + \frac{3\eta_i \eta_k}{\eta^4} \right) dS(\boldsymbol{\eta}) \doteq 3V_0 A^2 \delta_{ik}.$$

Applying the divergence theorem to $\mathscr{D}$, the region between S and S_∞, we can write

$$U_k \int_{S\infty} \left(\eta_i n_j \frac{\partial \tilde{\Phi}_k}{\partial \eta_j} - n_i \tilde{\Phi}_k \right) dS = U_k \int_S \left(\eta_i n_j \frac{\partial \tilde{\Phi}_k}{\partial \eta_j} - n_i \tilde{\Phi}_k \right) dS$$
$$= U_k \int_B \left(\eta_j n_j \frac{\partial \tilde{\Phi}_k}{\partial \eta_j} - n_i \tilde{\Phi}_k \right) dB + O(\epsilon),$$

where B is the surface of the pulsating body. The boundary conditions give

$$A U_k n_j \partial \tilde{\Phi}_k / \partial \eta_j = U_k n_k \quad \text{on } B$$

and hence

$$U_k \int_{S\infty} (\eta_i n_j \partial \tilde{\Phi}_k / \partial \eta_j - n_i \tilde{\Phi}_k) \, dS(\boldsymbol{\eta}) = U_k \int_B (A^{-1} \eta_i n_k - n_i \tilde{\Phi}_k) \, dB + O(\epsilon)$$
$$= \delta_{ik} A^2 V_0 U_k + \alpha_{ik} A^2 V_0 U_k + O(\epsilon),$$

where α_{ik} is the virtual mass tensor, defined by

$$V_0 A^3 \alpha_{ik} = -\int_B \Phi_k n_i \, dB.$$

We have now shown that

$$\frac{x_i}{r^2(1-M_r)^2} \frac{\epsilon^3 c_0}{a} \frac{dV}{dT} \int_{S\infty} \left\{ \eta_i n_j \frac{\partial \Theta}{\partial \eta_j} - n_i \Theta \right\} dS = \frac{3V_0 a \epsilon^3 c_0}{(1-M_r)^2} \frac{x_i}{r^2} \frac{dV}{dT} (2\delta_{ik} + \alpha_{ik}) U_k. \quad (10)$$

Finally, returning to a dimensional form and substituting (9) and (10) in (8) we obtain the form of the distant acoustic field:

$$4\pi P(\mathbf{x},t) = \frac{3V_0 a^2 \dot{v}}{r(1-M_r)}\left\{1+\frac{U_k}{c_0}\frac{x_i}{r}\frac{(2\delta_{ik}+\alpha_{ik})}{1-M_r}\right\}$$
$$= \frac{3V_0 a^2 \dot{v}}{r(1-M_r)^3}\left\{1+\frac{x_i}{r}\alpha_{ik}M_k\right\}+O(M^2). \qquad (11)$$

This is the main result of this paper.

It is surprising on two counts. The pulsating body is often regarded as the fundamental model of monopole sources, and monopole fields are generally thought to display a simple $(1-M_r)^{-2}$ factor due to convection. We see that such an impression is false, because the unsteady force on the body that must accompany any mass (and momentum) displacement destroys any simple source type specification of the field. Second, and now not so surprising, the influence of motion depends entirely on the body shape as determined through the virtual mass tensor.

In particular, if $\mathbf{M}$ lies along an eigenvector of the symmetric tensor α_{ij}, or along the body's axis of symmetry if one exists, then

$$\alpha_{ij}M_j = \alpha M_i,$$

where α is the eigenvalue corresponding to the eigenvector $\mathbf{M}$. α is then the drag force on the body as a function of the rate of change of the inertia in the fluid the body has displaced. Equation (11) reduces to a simpler form

$$4\pi P(\mathbf{x},t) = 3V_0 a^2 \dot{v}(1-M_r)^{-(3+\alpha)}/r. \qquad (12)$$

In general of course the force generated by potential flow on an arbitrary-shaped body of variable volume is not parallel to the direction of motion, and that is the origin of the more complex structure in (11). The consequences of source motion are then quite different from any description of convective effects we have seen in the literature on the subject (Lowson 1965; Morse & Ingard 1968; Ffowcs Williams & Hawkings 1969).

Furthermore, the result demonstrates a feature that may be of considerable practical importance. The consequences of source motion are *not* in general negligible at 90° to the direction of motion, as is invariably apparent from previous studies of convected sources. They all deal with mathematically convenient source descriptions and they may have little relevance to reality. This feature may provide a clue to the so-far perplexing issue of how aircraft motion affects the noise heard at 90° to the flight path. There it is known that effects exist which are not accountable for by Doppler effects (Hoch & Hawkins 1973). The change may be due to the need to model sources more realistically. These conclusions are surprising enough that we ought to check the analytical scheme. This can be done in the general case but for simplicity we give below only alternative proofs of the result for the particular case when the body is a pulsating sphere.

We consider therefore a radially pulsating sphere of radius $A(t)$ moving with low subsonic velocity $\mathbf{U}$. The virtual mass tensor is $\alpha_{ij} = \frac{1}{2}\delta_{ij}$ and $V_0 = \frac{4}{3}\pi$. From (12),

$$P(\mathbf{x},t) = \dot{v}a^2(1-M_r)^{-3\frac{1}{2}}/r,$$

a being the mean radius of the sphere, and in the linear sound field

$$P(\mathbf{x}, t) = (p - p_0)/\rho_0,$$

so that the far-field acoustic pressure is

$$(p - p_0)(\mathbf{x}, t) = \rho_0 a^2 \dot{v}(1 - M_r)^{-3\frac{1}{2}}/r.$$

We now obtain this result by two other methods.

3. Matched asymptotic expansions

In a frame in which the sphere is at rest and the fluid at infinity has a velocity $-\mathbf{U}$, the velocity potential ϕ satisfies the convected wave equation in the far field. The outer solution for ϕ is therefore of the form

$$\phi = -U\eta_1 + \frac{1}{\eta} f\left(t - \frac{\eta + M\eta_1}{c_0}\right) + \frac{\partial}{\partial \eta_i}\left(\frac{1}{\eta} g_i \left(t - \frac{\eta + M\eta_1}{c_0}\right)\right) + \dots .$$

In the inner field, the flow is by definition of the term 'inner' incompressible, so that $\nabla^2 \phi = 0$. An application of the boundary conditions on the sphere gives

$$\phi = -U\eta_1 - \frac{A^2 v(t)}{\eta} + \frac{\partial}{\partial \eta_1}\left(\frac{\frac{1}{2}UA^3(t)}{\eta}\right)$$

as the inner solution. By matching this to the outer solution we obtain a uniformly valid expression for the problem:

$$\phi(\boldsymbol{\eta}, t) = -U\eta_1 - \frac{A^2}{\eta} v\left(t - \frac{\eta + M\eta_1}{c_0}\right) + \frac{\partial}{\partial \eta_1}\left(\frac{\frac{1}{2}UA^3}{\eta}\left(t - \frac{\eta + M\eta_1}{c_0}\right)\right).$$

In the far field, after linearizing in v, this becomes

$$\phi(\boldsymbol{\eta}, t) = -U\eta_1 - \frac{a^2}{\eta} v\left(t - \frac{\eta + M\eta_1}{c_0}\right) - \frac{3Ua^2}{2c_0\,\eta} v \cos\bar{\theta},$$

where $\cos\bar{\theta} = \eta_1/\eta$. This distant potential is related to the pressure by

$$p(\boldsymbol{\eta}, t) - p_0 = -\rho\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial \eta_1}\right)\phi = \frac{\rho_0 \dot{v} a^2}{\eta}(1 + \tfrac{5}{2}M\cos\bar{\theta}).$$

For a frame $(\mathbf{x}, t)$ in which the fluid is at rest and the sphere moving

$$|\mathbf{x}| = \eta/(1 - M\cos\theta), \quad M\cos\bar{\theta} = M\cos\theta + O(M^2),$$

so that $$\phi(\mathbf{x}, t) = -a^2 v(\tau^*)(1 - M\cos\theta)^{-2\frac{1}{2}}/r$$

and $$(p - p_0)(\mathbf{x}, t = a^2 \dot{v}(\tau^*)(1 - M\cos\theta)^{-3\frac{1}{2}}/r,$$

as we have already determined from the general expression (11).

4. The Lighthill theory

The same result can be obtained from Lighthill's equation. We take the surface S to be the surface of a sphere moving with a velocity $\mathbf{U}$ and of fixed radius $a + \epsilon ka$, where k is large enough for the surface to enclose the pulsating sphere at all times. This surface S is not impenetrable so we have additional terms to the usual representation, the relevant form being

$$4\pi c_0^2(\rho-\rho_0) = \frac{\partial^2}{\partial x_i\,\partial x_j}\int_V \left[\frac{T_{ij}}{r(1-M_r)}\right] d\boldsymbol{\eta} - \frac{\partial}{\partial x_i}\int_S \left[\frac{p_{ij}+\rho u_i(u_j-U_j)}{r(1-M_r)}\right] n_j\, dS(\boldsymbol{\eta}) + \frac{\partial}{\partial t}\int_S \left[\frac{\rho_0 u_n}{(1-M_r)\,r}\right] dS(\boldsymbol{\eta}),$$

where T_{ij} is the Lighthill stress tensor $\rho u_i u_j + p_{ij} + c_0^2(\rho-\rho_0)\,\delta_{ij}$, p_{ij} is the compressive stress tensor, $M_r = \mathbf{U}.\mathbf{x}/(c_0|\mathbf{x}|)$, $\boldsymbol{\eta} = \mathbf{y} - \mathbf{U}t$ and square brackets imply evaluation at retarded time.

Again, near the body potential flow determines the source terms with an error of M^2 in comparison with unity. Hence in evaluating the source terms we use

$$\phi = -vA^2/\eta - \tfrac{1}{2}UA^3\eta_1/\eta^3.$$

The monopole term is

$$\frac{\partial}{\partial t}\int_S \rho_0\left[v + U\cos\overline{\theta}\,\frac{A^3}{(a+\epsilon)^3}\right]\frac{dS(\boldsymbol{\eta})}{r(1-M_r)} + \text{products of } v.$$

Expanding the retarded time gives

$$\frac{\partial}{\partial t}\int_S \left[\frac{\rho_0 u_n}{r(1-M_r)}\right] dS(\boldsymbol{\eta}) = \frac{\partial}{\partial t}\left\{\frac{4\pi a^2\rho_0 v}{r(1-M_r)} + \frac{M_r 4\pi a^2 \rho_0 v}{r(1-M_r)}\right\}$$

$$+ \text{higher time derivatives and products of } \mathbf{v}$$

$$= \rho_0 \dot{v} 4\pi a^2 (1-M_r)^{-3}/r.$$

The dipole term may be evaluated by first substituting for the pressure from Bernoulli's equation, then performing the integration to give

$$-\frac{\partial}{\partial x_i}\int_S \left[\frac{p_{ij}+\rho u_i(u_j-U_j)}{r(1-M_r)}\right] n_j\, dS = -\frac{\partial}{\partial x_1}\left\{\frac{2\pi\rho_0 a^2 U v}{r(1-M_r)}\right\} = \frac{2\pi\rho_0 a^2 M_r \dot{v}}{r(1-M_r)^2}.$$

Also we note that

$$\frac{\partial^2}{\partial x_i\,\partial x_j}\int_V \left[\frac{\rho u_i u_j + p_{ij} - c_0^2(\rho-\rho_0)\,\delta_{ij}}{r(1-M_r)}\right] = O(M^2\dot{v})\rho_0 a^2/r.$$

Hence

$$p - p_0 = \rho_0 a^2 \dot{v}(1-M_r)^{-3\frac{1}{2}}/r.$$

5. The sound field produced by a juddering compact body

We consider a rigid body of volume V_0 moving through a fluid at rest with a velocity $\mathbf{U}$, where $\mathbf{U}$ contains a small variable perturbation about a constant velocity, i.e. $\mathbf{U} = \mathbf{U}_0 + \epsilon\mathbf{v}(t)$, where U_0 is a constant and ϵ is a small parameter. We now non-dimensionalize velocities with respect to c_0, time with respect to $a/(c_0\epsilon)$, and introduce a non-dimensional velocity $\mathbf{V}$ and a non-dimensional slow time T defined by $\mathbf{V} = \mathbf{v}/c_0$ and $T = \epsilon c_0 t/a$. As a compactness condition we require that

$$\mathbf{V},\ d\mathbf{V}/dT,\ d^2\mathbf{V}/dT^2,\ \ldots,\ = O(1).$$

Hence
$$\mathbf{U} = c_0\mathbf{M} + \epsilon c_0 \mathbf{V},$$
$$d\mathbf{U}/dt = \epsilon^2 c_0^2 a^{-1} d\mathbf{V}/dT, \quad d^2\mathbf{U}/dt = \epsilon^3 c_0^3 a^{-2} d\mathbf{V}/dT.$$

M^2 is neglected in comparison with unity and again we may take the 'source' terms in (3) (cf. Howe 1975) to have the values determined by an incompressible flow with the same boundaries. As before, the incompressible flow is potential, and the far acoustic field is given by

$$4\pi(c_0^2(\rho/\rho_0 - 1) + \tfrac{1}{2}u^2) = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_V \left[\frac{\frac{1}{2}u^2}{r(1-M_r)}\right] d\boldsymbol{\eta} + \frac{\partial}{\partial x_i}\int_S \left[\frac{\partial\phi/\partial t}{r(1-M_r)}\right]_{\tau^*} n_i\, dS(\eta)$$
$$+ \int_S \left[\frac{\partial^2\phi/\partial y_i\,\partial t}{r(1-M_r)}\right]_{\tau^*} n_i\, dS(\boldsymbol{\eta}) + Q(\mathbf{x}, t),$$

where
$$\tau^* = \frac{1}{1-M_r}\left(t - \frac{|\mathbf{x}|}{c_0} + \frac{\mathbf{x}\,.\,\boldsymbol{\eta}}{|\mathbf{x}|\,c_0}\right),$$

S is the surface of the body and V is the volume of fluid outside S.

We can combine the dipole and monopole surface terms in this equation, and a Taylor expansion of retarded time gives

$$\frac{\partial}{\partial x_i}\int_S \left[\frac{\partial\phi/\partial t}{r(1-M_r)}\right]_{\tau^*} n_i\, dS(\boldsymbol{\eta}) + \int_S \left[\frac{\partial^2\phi/\partial y_i\,\partial t}{r(1-M_r)}\right]_{\tau^*} n_i\, dS(\boldsymbol{\eta})$$
$$= \frac{1}{r(1-M_r)}\int_S \left(\frac{\partial^2\phi}{\partial y_i\,\partial t} n_i\right)_{\tau_1} dS(\boldsymbol{\eta}) + \frac{x_i}{r^2(1-M_r)\,c_0}$$
$$\times \frac{\partial}{\partial\tau}\left\{\frac{1}{1-M_r}\int_S \left(\eta_i \frac{\partial^2\phi}{\partial y_j\,\partial t} n_j - n_i \frac{\partial\phi}{\partial t}\bigg|_{\mathbf{y}}\right) dS(\boldsymbol{\eta})\right\}_{\tau_1} + \frac{x_j x_k}{r^3 c_0^2(1-M_r)}$$
$$\times \frac{\partial}{\partial\tau}\left(\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left\{\frac{1}{1-M_r}\int_S \left(\tfrac{1}{2}\eta_j\eta_k n_i \frac{\partial^2\phi}{\partial y_i\,\partial t} - n_j\eta_k\frac{\partial\phi}{\partial t}\right) dS(\boldsymbol{\eta})\right\}\right)_{\tau_1} + O(\epsilon^4), \tag{13}$$

where
$$\tau_1 = \frac{1}{1-M_r}\left(t - \frac{|\mathbf{x}|}{c_0}\right)$$

and three terms of the retarded time expansion in the monopole and two terms in the dipole give terms up to order ϵ^3.

The velocity potential ϕ satisfies $\nabla^2\phi = 0$ in V with the boundary condition $\partial\phi/\partial n = \mathbf{U}\,.\,\mathbf{n}$ on S. It is well known from potential theory (e.g. Batchelor 1970, p. 403) that such a ϕ can be written as $\phi = \mathbf{U}\,.\,\boldsymbol{\Phi}(\boldsymbol{\eta})$, where

$$\Phi_i = -V_0(\alpha_{ij} + \delta_{ij})\,\eta_j/4\pi\eta^3 + \text{higher harmonics}. \tag{14}$$

Then
$$\partial\phi/\partial t|_{\mathbf{y}} = \dot{\mathbf{U}}\,.\,\boldsymbol{\Phi} - \mathbf{U}\,.\,\nabla\phi. \tag{15}$$

$\boldsymbol{\Phi}$ is a well-defined harmonic function, and the integrals in (13) may be simplified by applying the divergence theorem to the region $\mathscr{D}$ between the surface S and a large sphere S_∞; again there is no implication that $\boldsymbol{\Phi}$ represents the actual distant field. Then

$$\int_S \frac{\partial^2\phi}{\partial y_i\,\partial t} n_i\, dS(\boldsymbol{\eta}) = \int_{S_\infty} \frac{\partial^2\phi}{\partial y_i\,\partial t} n_i\, dS(\boldsymbol{\eta}) - \int_{\mathscr{D}} \frac{\partial^3\phi}{\partial\eta_i\,\partial y_i\,\partial t} d^3\boldsymbol{\eta}$$
$$= 0$$

since $\phi = O(\eta^{-2})$ on S_∞ and $\partial^3\phi/\partial\eta_j\,\partial y_j\,\partial t = \partial^3\phi/\partial y_j^2\,\partial t = 0$ in $\mathscr{D}$. Substituting for $\partial\phi/\partial t|_{\mathbf{y}}$ from (15) we obtain

$$\int_S \left(\eta_i \frac{\partial^2\phi}{\partial y_j\,\partial t} n_j - n_i \frac{\partial\phi}{\partial t}\right) dS = \dot{U}_k \int_S \left(\eta_i \frac{\partial\Phi_k}{\partial y_j} n_j - n_i\,\Phi_k\right) dS$$
$$- U_m U_l \int_S \left(\eta_i \frac{\partial^2\Phi_m}{\partial y_j\,\partial y_l} n_j - n_i \frac{\partial\Phi_m}{\partial y_l}\right) dS.$$

The boundary condition on the body is $n_j\,\partial\Phi_k/\partial y_i = n_k$. Hence, after applying the divergence theorem to the second integral

$$\int_S \left(\eta_i \frac{\partial^2\phi}{\partial y_j\,\partial t} n_j - n_i \frac{\partial\phi}{\partial t}\right) dS = \dot{U}_k \int_S (\eta_i n_k - n_i\,\Phi_k)\, dS$$
$$- U_m U_l \int_{S_\infty} \left(\eta_i \frac{\partial^2\Phi_m}{\partial y_j\,\partial y_l} n_j - n_i \frac{\partial\Phi_m}{\partial y_l}\right) dS.$$

All the integrals on the right-hand side can now be evaluated, giving

$$\int_S \left(\eta_i \frac{\partial^2\phi}{\partial y_j\,\partial t} n_j - n_i \frac{\partial\phi}{\partial t}\right) dS = \dot{U}_k V_0(\delta_{ik} + \alpha_{ik}). \tag{16}$$

We now apply the same technique to the next integral in (13). Substituting for $\partial\phi/\partial t|_{\mathbf{y}}$ gives

$$\int_S \left(\tfrac{1}{2}\eta_j\eta_k n_i \frac{\partial^2\phi}{\partial y_i\,\partial t} - n_j\eta_k \frac{\partial\phi}{\partial t}\right) dS = \tfrac{1}{2}U_l U_m \int_S \left(n_j\eta_k \frac{\partial\Phi_m}{\partial y_l} - \eta_j\eta_k n_i \frac{\partial^2\Phi_m}{\partial y_i\,\partial y_l}\right) dS$$
$$+ \tfrac{1}{2}U_l U_m \int_S n_j\eta_k \frac{\partial\Phi_m}{\partial y_l} dS + O(\epsilon U^2 V_0).$$

Applying the divergence theorem to the first integral on the right-hand side yields

$$\int_S \left(\tfrac{1}{2}\eta_j\eta_k n_i \frac{\partial^2\phi}{\partial y_i\,\partial t} - n_j\eta_k \frac{\partial\phi}{\partial t}\right) dS$$
$$= \tfrac{1}{2}U_l U_m \left\{ \int_{S_\infty} \left(n_j\eta_k \frac{\partial\Phi_m}{\partial y_l} - \eta_j\eta_k n_i \frac{\partial^2\Phi_m}{\partial y_i \partial y_l}\right) dS \right.$$
$$- \int_{\mathscr{D}} \left(\delta_{jk} \frac{\partial\Phi_m}{\partial y_l} + \eta_k \frac{\partial^2\Phi_m}{\partial\eta_j\,\partial y_l} - \delta_{ij}\eta_k \frac{\partial^2\Phi_m}{\partial\eta_i\,\partial y_l} - \delta_{ik}\eta_j \frac{\partial^2\Phi_m}{\partial\eta_i\,\partial y_l}\right) d^3\boldsymbol{\eta}$$
$$\left. + \int_S n_j\eta_k \frac{\partial\Phi_m}{\partial y_l} dS \right\} + O(\epsilon U^2 V_0)$$
$$= \tfrac{1}{2}U_l U_m \int_{S_\infty} \left\{(n_j\eta_k + \eta_j n_k) \frac{\partial\Phi_m}{\partial\eta_l} - \eta_j\eta_k n_i \frac{\partial^2\Phi_m}{\partial\eta_i\,\partial\eta_l} - 2\delta_{jk}\,\Phi_m\, n_l\right\} dS$$
$$+ \tfrac{1}{2}U_l U_m \int_S \left\{2\delta_{jk}\,\Phi_m n_l + \frac{\partial\Phi_m}{\partial y_l}(\eta_k n_j - \eta_j n_k)\right\} dS + O(\epsilon U^2 V_0) \tag{17}$$

by a second application of the divergence theorem. The integral over S_∞ may be evaluated using the far-field form of Φ_m given in (14) and the two standard integrals

$$\int_{\text{sphere}} \eta^{-2} n_j n_k\, dS = \tfrac{4}{3}\pi\delta_{jk}, \quad \int_{\text{sphere}} \eta^{-2} n_i n_j n_k n_l\, dS = \tfrac{4}{15}\pi\{\delta_{ij}\delta_{kl} + \delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}\}$$

to give

$$\int_{S_\infty}\left\{(n_j\eta_k+\eta_j n_k)\frac{\partial\Phi_m}{\partial\eta_l}-\eta_j\eta_k n_i\frac{\partial^2\Phi_m}{\partial\eta_i\,\partial\eta_l}-2\delta_{jk}\,\Phi_m\,n_l\right\}dS$$
$$=V_0(\delta_{mi}+\alpha_{mi})\{\delta_{ij}\delta_{kl}+\delta_{jl}\delta_{ik}\}. \quad (18)$$

We can simplify the integral over S in (17) because

$$U_l U_m\int_S\left\{2\delta_{jk}\,\Phi_m n_l+\frac{\partial\Phi_m}{\partial y_l}(\eta_k n_j-\eta_j n_k)\right\}dS=-2\delta_{jk}\,U_m U_l\alpha_{ml}V_0$$
$$+\text{ terms antisymmetric in } j \text{ and } k. \quad (19)$$

Substituting (18) and (19) into (17), we have

$$\int_S(\tfrac{1}{2}\eta_j\eta_k n_i\,\partial^2\phi/\partial y_i\,\partial t-n_i\eta_k\,\partial\phi/\partial t)\,dS$$
$$=-\delta_{jk}U_m U_l\alpha_{ml}V_0+\tfrac{1}{2}U_m U_l V_0(\delta_{jm}\delta_{kl}+\delta_{jl}\delta_{mk}+\alpha_{mj}\delta_{kl}+\alpha_{mk}\delta_{jl})$$
$$+\text{ terms antisymmetric in } j \text{ and } k\ +O(\epsilon U^2V_0)$$

and from (13)

$$\frac{\partial}{\partial x_i}\int_S\left[\frac{\partial\phi/\partial t}{r(1-M_r)}\right]n_i\,dS(\boldsymbol{\eta})+\int_S\left[\frac{\partial^2\phi/\partial y_i\,\partial t}{r(1-M_r)}\right]n_i\,dS(\boldsymbol{\eta})$$
$$=\frac{x_i\epsilon^3c_0^2V_0}{r^2(1-M_r)^2a^2}(\alpha_{ij}+\delta_{ij})\frac{d^2V_j}{dT^2}+\frac{x_jx_kV_0\epsilon^2c_0^2}{r^3(1-M_r)^3a^2}\left\{2M_j\frac{d^2V_k}{dT^2}+\alpha_{jm}M_k\frac{d^2V_m}{dT^2}\right.$$
$$\left.+\alpha_{jm}M_m\frac{d^2V_k}{dT^2}-2\delta_{jk}\alpha_{ml}M_l\frac{d^2V_m}{dT^2}\right\}+O(\epsilon^4c_0^2a^{-2}V_0).$$

We have only the volume source term in (4) still to evaluate.

The total volume source strength is

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_V\tfrac{1}{2}u^2\,dV+O(\epsilon^4c_0^2a^{-2}V_0)+O(\epsilon^3M^2c_0^2a^{-2}V_0)$$
$$=\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_V\tfrac{1}{2}\nabla\phi\,.\,\nabla\phi\,dV+O(\epsilon^4c_0^2a^{-2}V_0)+O(\epsilon^3M^2c_0^2a^{-2}V_0)$$
$$=-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_S\tfrac{1}{2}\phi\nabla\phi\,.\,\mathbf{n}\,dS+O(\epsilon^4c_0^2a^{-2}V_0)+O(\epsilon^3M^2c_0^2a^{-2}V_0)$$
$$=-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_S\tfrac{1}{2}\phi\mathbf{U}\,.\,\mathbf{n}\,dS+O(\epsilon^4c_0^2a^{-2}V_0)+O(\epsilon^3M^2c_0^2a^{-2}V_0)$$
$$\approx\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}U_lU_m\alpha_{lm}V_0)$$

by the definition of α_{lm}. Let S_L be a large sphere of radius R_L, and V_L be the region outside this sphere. Then the total source strength in V_L is

$$\int_{V_L}\tfrac{1}{2}u^2\,dV=\int_{S_L}\phi\nabla\phi\,.\,\mathbf{n}\,dS=O(U^2V_0^2R_L^{-3})\quad\text{from the far-field form of }\boldsymbol{\Phi}\text{ in (14)}$$
$$=O(U^2V_0\epsilon)\text{ if }R_L\geqslant(V_0\epsilon^{-1})^{\frac{1}{3}}.$$

Hence the total source strength outside a sphere of radius $(V_0\epsilon^{-1})^{\frac{1}{3}}$ is of order ϵ smaller than the source strength within such a sphere. Retaining only the lowest-order terms in ϵ, we have a *finite* source region, and

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int_V\left[\frac{\frac{1}{2}u^2}{r(1-M_r)}\right]d^3\eta = \frac{c_0^2\epsilon^3}{a^2}\frac{V_0 M_l}{r(1-M_r)^3}\alpha_{lm}\frac{d^2V_m}{dT^2}+O(\epsilon^4c_0^2a^{-2}V_0)$$

for points in the far field such that $r \gg (V_0\epsilon^{-1})^{\frac{1}{3}}$. Returning to dimensional times, from (4) the total acoustic far field is described by

$$4\pi((p-p_0)/\rho_0+\tfrac{1}{2}u^2)(\mathbf{x},t)$$
$$= \frac{V_0(\delta_{rj}+\alpha_{rj})}{r(1-M_r)^2}\ddot{M}_j+\frac{V_0}{r(1-M_r)^3}\{2M_r\ddot{M}_r+\alpha_{rm}\ddot{M}_m M_r+\alpha_{rm}M_m\ddot{M}_r-M_l\alpha_{lm}\ddot{M}_m\}, \quad (20)$$

where the suffix r denotes the direction of the field variable $\mathbf{x}$, and $\dot{M}_l = dM_l/dt$; again M^2 has been neglected in comparison with unity. The first term is that due to a moving dipole, and it is usually taken to describe the acoustic field. We see that this is incorrect and that to describe the effect of convection we must retain the second term, which has a very complicated structure and contains an omni-directional term.

This form simplifies when $\ddot{\mathbf{M}}$ is parallel to $\mathbf{M}$ and both vectors lie along an eigenvector of the symmetric tensor α_{ij}. If α is the corresponding eigenvalue then

$$\alpha_{ij}M_j = \alpha M_i, \quad \alpha_{ij}\ddot{M}_j = \alpha\ddot{M}_i$$

and (20) has the simpler form

$$4\pi\left(\frac{p-p_0}{\rho_0}+\tfrac{1}{2}u^2\right) = \frac{V_0(1+\alpha)\ddot{M}_r}{r(1-M_r)^4}-\frac{V_0\alpha M\ddot{M}}{r}.$$

The first term on the right-hand side has the value of the acoustic field modified by motion by a Doppler factor to the fourth power. The last term represents a non-directional field, which produces amplification even at 90° to the motion.

6. Conclusion

We have seen that the convective amplification obtained from a moving pulsating body is different to that obtained from moving elementary sources. This is because a moving pulsating body, producing a mass flux, necessarily has associated with it a momentum flux. Hence the sound field produced is that of a moving monopole and a coupled dipole. Moreover, the strength of the dipole is smaller by only a factor of the order of the Mach number than the monopole and so affects the Doppler amplification factor.

Investigation here of the far-field form of the acoustic field produced by a moving pulsating body of arbitrary shape has determined that it is amplified from its value at rest by a factor

$$(1+\alpha_{rj}M_j)/(1-M_r)^3,$$

where α_{ij} is the virtual mass tensor. When $\mathbf{M}$ lies along an eigenvector of the symmetric tensor the amplifying factor is $(1-M_r)^{-(3+\alpha)}$, where α is the corresponding eigenvalue. However, when $\mathbf{M}$ is not in the direction of an eigenvector of α, the far field may be amplified even for points which lie in a plane perpendicular to the direction of motion.

The distant form of the acoustic field produced by the oscillation of a convected compact body has also been investigated. We have found that this field is *not* that of a moving dipole as is usually supposed. The effect of motion is complicated and quite unpredictable from any previous studies. It is not completely described in terms of Doppler factors, and contains an additional omnidirectional term of strength $-\ddot{M}_i\,\alpha_{ij}\,M_j\,V_0$.

When the motion is parallel to an eigenvalue of the symmetric tensor α_{ij}, the non-convected field is amplified by a factor $(1-M_r)^{-4}$, and there is an additiona omnidirectional field.

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Appendix

From (1) we have

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)P = -\nabla.(\mathbf{u}\wedge\boldsymbol{\omega})+\nabla^2\left\{\int\rho^{-1}dp-c_0^2(\rho/\rho_0-1)\right\}$$
$$+\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(\tfrac{1}{2}u^2)-\frac{\partial^2}{\partial x_i\,\partial t}\{(\rho/\rho_0-1)u_i\} \quad \text{(A 1)}$$

throughout the fluid, where $P=(\rho/\rho_0-1)c_0^2+\frac{1}{2}u^2$. We introduce a bounded closed surface S which encloses the body and possibly some fluid too. Then we define a function $f(\mathbf{x},t)$ by

$$\left.\begin{array}{lll} f(\mathbf{x},t)>0 & \text{for points } (\mathbf{x},t) \text{ outside} & S,\\ f(\mathbf{x},t)<0 & \text{for points } (\mathbf{x},t) \text{ within} & S,\\ f(\mathbf{x},t)=0 & \text{for points } (\mathbf{x},t) \text{ on} & S. \end{array}\right\} \quad \text{(A 2)}$$

The velocity $\mathbf{v}$ of the surface is such that

$$\partial f/\partial t+v_i\,\partial f/\partial x_i = 0. \quad \text{(A 3)}$$

By multiplying (A 1) by $H(f)$, where H is the Heaviside function, we obtain the global equation

$$H(f)\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)P = -H\nabla.(\mathbf{u}\wedge\boldsymbol{\omega})+H\nabla^2\left\{\int\rho^{-1}dp-c_0^2(\rho/\rho_0-1)\right\}$$
$$+H\frac{\partial^2}{\partial t^2}\left(\frac{\frac{1}{2}u^2}{c_0^2}\right)-H\frac{\partial^2}{\partial x_i\,\partial t}\{(\rho/\rho_0-1)\,u_i\}, \quad \text{(A 4)}$$

or alternatively

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)HP = -\frac{\partial}{\partial x_i}(H(\mathbf{u}\wedge\boldsymbol{\omega})_i)+H\nabla^2\left\{\int\rho^{-1}dp-c_0^2(\rho/\rho_0-1)\right\}+\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}H\frac{u^2}{c_0^2}\right)$$
$$-\frac{\partial^2}{\partial t\,\partial x_i}\{Hu_i(\rho/\rho_0-1)\}-\frac{\partial}{\partial x_i}\left\{\frac{\partial H}{\partial x_i}P\right\}+\frac{\partial}{\partial t}\left\{\frac{\partial H}{\partial x_i}(u_i-v_i)(\rho/\rho_0-1)\right\}$$
$$+\frac{\partial H}{\partial x_i}\left\{-\frac{\partial P}{\partial x_i}+(\mathbf{u}\wedge\boldsymbol{\omega})_i-v_i\left(\frac{\partial}{\partial t}(\rho/\rho_0)+\frac{\partial}{\partial x_j}\{u_j(\rho/\rho_0-1)\}\right)\right\}. \quad \text{(A 5)}$$

Since $\partial\rho/\partial t+\nabla.(\rho\mathbf{u}) = 0$, we can simplify the last term to obtain

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)HP = -\frac{\partial}{\partial x_i}(H(\mathbf{u}\wedge\boldsymbol{\omega})_i)+H\nabla^2\left\{\int\rho^{-1}dp-c_0^2(\rho/\rho_0-1)\right\}$$
$$+\frac{\partial^2}{\partial t^2}\left(\frac{Hu^2}{2c_0^2}\right)-\frac{\partial^2}{\partial t\,\partial x_i}\{Hu_i(\rho/\rho_0-1)\}$$
$$-\frac{\partial}{\partial x_i}\left\{\frac{\partial H}{\partial x_i}P\right\}+\frac{\partial}{\partial t}\left\{\frac{\partial H}{\partial x_i}(\rho/\rho_0-1)(u_i-v_i)\right\}$$
$$+\frac{\partial H}{\partial x_i}\left\{-\frac{\partial P}{\partial y_i}+(\mathbf{u}\wedge\boldsymbol{\omega})_i+v_i\nabla.\mathbf{u}\right\}. \quad \text{(A 6)}$$

As in the work by Ffowcs Williams & Hawkings (1969) we can write this as an integral equation:

$$4\pi H(\mathbf{x},t)P(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\int_V\left[\frac{J(\mathbf{u}\wedge\boldsymbol{\omega})_i}{r(1-M_r)}\right]d^3\boldsymbol{\eta}+\int_V\left[\frac{J\nabla^2\{\int\rho^{-1}dp-c_0^2(\rho/\rho_0-1)\}}{r(1-M_r)}\right]d^3\boldsymbol{\eta}$$
$$+\frac{\partial^2}{\partial t^2}\int_V\left[\frac{J\frac{1}{2}u^2/c_0^2}{r(1-M_r)}\right]d^3\boldsymbol{\eta}-\frac{\partial^2}{\partial t\,\partial x_i}\int_V\left[\frac{J(\rho/\rho_0-1)u_i}{r(1-M_r)}\right]d^3\boldsymbol{\eta}-\frac{\partial}{\partial x_i}\int_S\left[\frac{AP}{r(1-M_r)}\right]n_i\,dS(\boldsymbol{\eta})$$
$$+\frac{\partial}{\partial t}\int_S\left[\frac{A(\rho/\rho_0-1)(u_i-v_i)}{r(1-M_r)}\right]n_i\,dS(\boldsymbol{\eta})+\int_S\left[\frac{A(-\partial P/\partial y_i+(\mathbf{u}\wedge\boldsymbol{\omega})_i+v_i\nabla.\mathbf{u})}{r(1-M_r)}\right]$$
$$\times n_i\,dS(\boldsymbol{\eta}),$$

where $\boldsymbol{\eta}$ are moving co-ordinates such that S is a constant function of $\boldsymbol{\eta}$, $M_r = \mathbf{v}.\mathbf{x}/(c_0|\mathbf{x}|)$, J is the volume Jacobian of the transformation, A is the surface Jacobian of the transformation and square brackets mean 'evaluated at at retarded time'.

For our purposes here it is most convenient to take S to be a surface of fixed geometry which moves downstream with velocity $\mathbf{U}$. Then a suitable set of moving co-ordinates is $\boldsymbol{\eta} = \mathbf{y}-\mathbf{U}t$, giving $J = A = 1$.

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